Microwave Noise in Semiconductor Devices

Module 2

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1 Stochastic signals in time and frequency domain

[Ambrozy Chap 3, all figures from here]

I, V

We have defined a stochastic process as a single parameter assembly of random variables over a set of times. The RV, $\xi(t)$, could represent voltage or current and be regarded as a discrete or continuous function of time.

In many cases, the function describing e.g. voltage is given by a formula with a precise time dependence - a deterministic function. Often a few parameters - amplitude, frequency, etc - fully specify the past and future of the signal over some time interval.

A stochastic signal does not allow knowledge of the next instantaneous value! Only probabilities. In practice, some correlation might exist owing to small but finite time constants (e.g. transit time of an electron across a gap). For very small time intervals, the instantaneous values are not independent of each other. This property is known as autocorrelation.

First, we need to describe deterministic and stochastic signals in the time and frequency domain.

2 Deterministic and stochastic signals

Simplest deterministic signal - cosines and sines with integral multiples of frequency

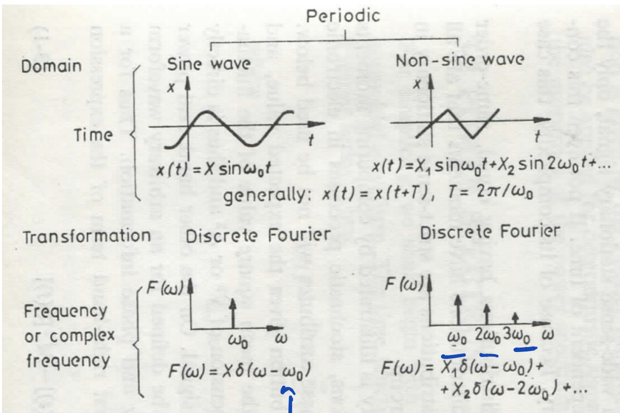


Figure 1: Schematic of various types of signals.

fundamental

To get to frequency domain → discrete Fourier transform. → fourier series

Almost periodic signal - superposition of sines and cosines

with various frequencies satisfying ω_j/ω_k = irrational #

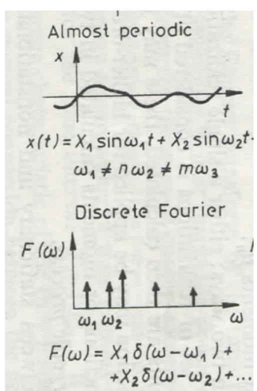


Figure 2: Almost periodic signals.

Such a function repeats only after infinite time. They also have a discrete frequency domain spectrum.

Transients - functions for which $x(t) = 0$ if $t < 0$. We use a Fourier transform to get a continuous frequency spectrum. Although the Fourier transform might not converge, we use it with caution anyway.

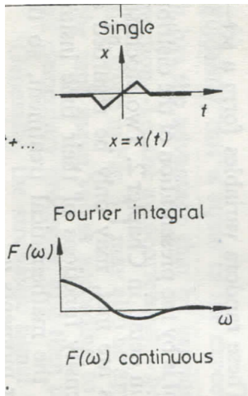


Figure 3: Continuous signal and its Fourier transform.

Stochastic signals are similar to almost periodic signals, but the frequency spectrum of stochastic signals is continuous.

Since we can't predict the amplitude and phase of the signal at an arbitrary time instant, we also don't know the amplitude and phase of the Fourier transform. We need another description.

Let's figure it out. We can define a stochastic process by a function $\xi(t)$, just as for a deterministic process. $\xi(t)$ may have real or complex values (or vectors). If we get a specific function $\xi(t)$ during a trial, that is called a realization of the stochastic process.

On the other hand, say we fix time t. $\xi(t)$ may have different values during consecutive trials so that $\xi(t)$ at fixed t is also a random variable. This set of random variables is called an ensemble.

We can characterize a stochastic signal by its distribution function (as in Module 1). We can say two stochastic functions are equivalent if the ensemble density functions are equal. Often in place of density functions we use their moments M_n.

So we have two ways to define a density function for a stochastic process:

a single realization as t varies (first moment = time average);

set of realizations at a fixed time instant (first moment = set average);

We can do the same for higher-order moments.

Ergodic process - the case in which the density functions (or moments) calculated these two different ways are the same.

With this way to characterize stochastic processes, we can define

stationary and non-stationary stochastic signals.

Stationary | Non-stationary

$\mu_n \neq \mu_n(t)$ | $\mu_n = \mu_n(t)$

$\uparrow$ $(f(t))$

All moments independent of time.

Ergodic group: contains distributions having equal, same-order moments. Most often, distributions in electronic devices are stationary and ergodic.

Now let's consider the physical meaning of the moments.

First-order - expected value $M(\xi)$

Central second-order - mean square value of fluctuation → variance.

$\langle(\xi - \langle\xi\rangle)^2\rangle$

Dimension of second-order moment = V^2 or A^2 → connected to power of the stochastic signal.

Real power is a physical quantity that can be converted to heat for any waveform (frequency and phase information is lost). So the power spectrum - or power contained in various frequency bands - is a physically meaningful and measurable quantity.

Later we will see the link between the time-domain equations and power spectrum is via Fourier transform and autocorrelation function. The ACF is:

$$\Gamma_\zeta(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \zeta(t)\,\zeta(t+\tau)\,dt$$

This function is the expected value of $\zeta^2(t)$ for $\tau=0$. It can also be regarded as the ~~DC component of the~~ total average power power in the signal. For $\tau \neq 0$, it also shows the correlation between consecutive time instants.

Intuitively, we should have that the total power in the range $0<f<\infty$ ← freq. domain must equal the power obtained from the mean square value of time instants for $T \gg f_{min}^{-1}$. We will derive Parseval's theorem that states this fact soon.

2.1 Power spectrum

Consider a deterministic (periodic non-sinusoidal) waveform. We can always expand it in a Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right)$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

$$T = \frac{2\pi}{\omega_0} = \text{fundamental period}$$

Using the Euler relation, we get:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + jb_n}{2}e^{-jn\omega_0 t} + \frac{a_n - jb_n}{2}e^{jn\omega_0 t}\right)$$

Introducing complex coefficients:

$$c_n^- = \frac{a_n + jb_n}{2} \qquad c_n^+ = \frac{a_n - jb_n}{2}$$

So we get $x(t)$ consisting of a constant and vectors rotating at $\pm n\omega_0$.

Take $a_0 = 0$. Note that $C_n^- = (C_n^+)^*$

For negative frequencies, note that b_n switches sign, but a_n does not. So for all , we can use the universal coefficient:

$$C_n = \frac{a_n - j b_n}{2} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

rather than separate coefficients C_n^- and C_n^+ . So the original signal can be expanded as:

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

With this expansion, we can obtain the mean square value of $x(t)$. We multiply the Fourier expansion of $x(t)$ by its complex conjugate and average in time:

$$M[x^2(t)] = \frac{1}{T}\int_{-T/2}^{T/2} \left(C_1 C_1^* + C_1 e^{j\omega_0 t} C_2^* e^{-j2\omega_0 t} + \ldots + C_2 C_2^* \ldots\right) dt$$
$$= \frac{1}{T}\int \sum_{n,m} C_n C_m^* e^{j(n-m)\omega_0 t}\, dt$$

The period $T = 2\pi/\omega_0$. For $n \neq m$, the integral term gives zero.

So we get: $$M[x^2(t)] = \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Also remember that C_n^+ and C_n^- for the same n are conjugates and hence have equal magnitudes. Therefore, $|C_n|^2$ is even.

Here is an example plot of the Fourier coefficients and double-sided power spectrum versus frequency:

$-\infty < f < \infty$

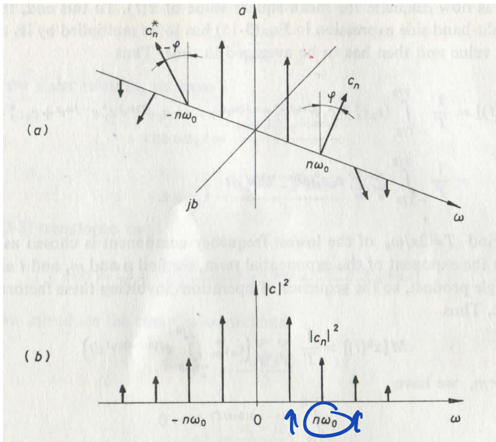


Figure 3-2 Periodic nonsinusoidal signal. *(a)* Complex Fourier coefficients as a function of discrete $n\omega_0$ frequencies. *(b)* Double-sided power spectrum.

Figure 4:

Note that the dimension of c_n is A or V, and so the dimension of $|c_n|^2$ is A^2 or V^2. If the given voltage (or current) is dropped across (or flows through) a resistance R, we see we get units of power → we have a power spectrum:

$$P = R\sum_{n=-\infty}^{\infty}|c_n|^2 \quad [W], \quad i = \text{current}$$

Often we just refer to $|c_n|^2$ as the power spectrum.

So we get the theorem: the power of a periodic nonsinusoidal signal can be computed from the Fourier coefficients.

Regarding the negative frequencies: since $|c_n|$ is even and we have assumed no DC offset ($c_0 = a_0 = 0$), we have:

$$M[x^2(t)] = 2\sum_{n=1}^{\infty}|c_n|^2 = \sum_{n=1}^{\infty}|C_n|^2$$

Notation: lower case = two sided power spectrum, upper case = one sided power spectrum (only positive frequencies).

Now consider a deterministic signal with a continuous Fourier spectrum. The discrete spectrum → continuous spectrum $C_n \to F(\omega)$, using:

$$F(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

The function $x(t)$ must satisfy certain requirements.

1. $x(t)$ should be continuous and differentiable.
2. Integral of absolute value of $x(t)$ should be finite:

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \text{finite} \quad \to \quad \begin{array}{l} t \to \pm\infty \\ x(t) \to 0. \end{array}$$

Since (2) is quite restrictive, we can instead require that $x(t)$ is defined on $(-T, T)$:

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{outside} \end{cases}$$

So that the Fourier transform is:

$$F_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

Example: ramp function $x(t)$

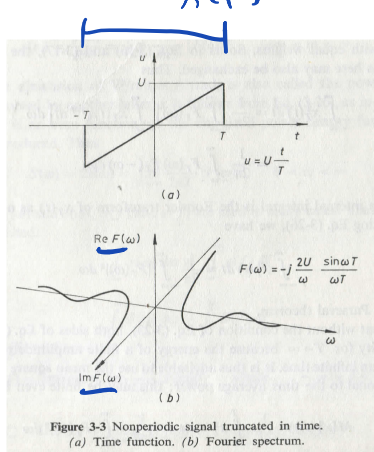


Figure 3-3 Nonperiodic signal truncated in time. *(a)* Time function. *(b)* Fourier spectrum.

Figure 5: Ramp function.

Some other nice properties.

Since $x(t)$ is real (e.g. it consists of voltage values from a scope), we can write:

$$F_T(\omega) = \int_{-\infty}^{\infty} x_T(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} x_T(t) \sin \omega t \, dt$$

Now rewrite with $-\omega$ rather than ω. First term is the same, second term changes sign. So

$$F_T(-\omega) = F_T^*(\omega)$$

If we know $F_T(\omega)$, we can get $x_T(t)$ by inverse Fourier transform:

$$x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_T(\omega) \, e^{j\omega t} \, d\omega$$

Now let's do something more interesting by computing the time integral of x_T^2. This is proportional to the energy in the signal:

$$\int_{-\infty}^{\infty} x_T^2(t) \, dt = \int_{-\infty}^{\infty} x_T(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F_T(\omega) \, e^{j\omega t} \, d\omega \right] dt$$

We can switch the order of integrations:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_T(\omega) \left[\int_{-\infty}^{\infty} x_T(t) \, e^{j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_T(\omega) \, F_T(-\omega) \, d\omega$$

And using the relation at top of page, we get Parseval's theorem:

$$\int_{-\infty}^{\infty} x_T^2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 \, d\omega$$

A subtlety: for signals that have a finite value outside of a time window (e.g. NOT the ramp function shown before), the ~~power~~ energy can $\to \infty$ as $T \to \infty$. We can instead compute the mean square value = time average power:

$$M\left[x_T^2(t)\right] = \frac{1}{2T}\int_{-T}^{T} x_T^2(t)\,dt = \frac{1}{4\pi T}\int_{-\infty}^{\infty} |F_T(\omega)|^2\,d\omega$$

Considering functions without the time window restriction (just requiring that $M[x_T^2(t)]$ goes to a finite limiting value for $T \to \infty$):

$$M\left[x^2(t)\right] = \lim_{T\to\infty} \frac{1}{4\pi T}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Taking $x(t)$ to have units of current or voltage, average power is:

current

$$\underline{P} = R\, M\left[x^2(t)\right]$$

voltage

$$\underline{P} = \frac{M\left[x^2(t)\right]}{R}$$

From here we can define a power density:

$$p(\omega) = \frac{d\underline{P}}{d\omega} = R \lim_{T\to\infty} \frac{1}{4\pi T} |F(\omega)|^2$$

Note: this definition is for a two-sided power spectrum: $-\infty < \omega < \infty$.

Often another quantity is also referred to as the power density:

$$s(\omega) \underset{\substack{\uparrow \\ \text{current}}}{=} \frac{p(\omega)}{R} = \lim_{T\to\infty} \frac{1}{4\pi T} |F(\omega)|^2$$

Units: V^2/rad or A^2/rad

Since $|F(\omega)|^2$ is an even function, we can introduce the one-sided power density:

$$S(\omega) = 2\, s(\omega)$$

Finally, we often express the distribution functions in terms of f rather than ω. Using $d\omega = 2\pi df$, we get:

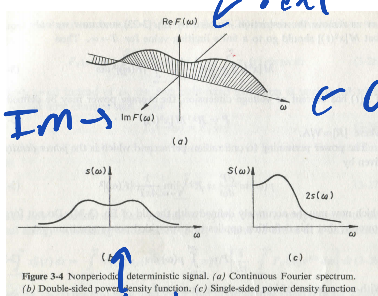
$$s(f) = 2\pi s(\omega) \qquad (-\infty < f < \infty)$$

$$S(f) = 2\pi S(\omega) \qquad (0 < f < \infty)$$

Here is an example of the Fourier spectrum of a signal and its power spectra:

← real

Im →



← ω

← single-sided

double

Figure 6: Example Fourier spectrum.

Note that $F(\omega)$ is a complex function → contains 2 real numbers for each ω! However, $s(\omega)$ has only one real number. $F(\omega)$ cannot be determined from $s(\omega)$.

What happens for stochastic processes? We can still define a mean square value and hence obtain the average power.The following result still holds:

$$\rightarrow \qquad M[\xi^2(t)] = \int_{-\infty}^{\infty} s(\omega)\, d\omega$$

The problem that arises for stochastic signals is that $F(\omega)$ may not exist. Consider a stationary signal of infinite time duration. Condition (2) is not met since:

$$\int_{-\infty}^{\infty} |\xi(t)|\, dt \rightarrow \infty$$

Interestingly, $s(\omega)$ can still be defined. The stationarity of the stochastic process causes

$$M[\xi^2(t)] = \mu_2[\xi(t)]$$

to be a finite, time-independent constant. That means that

$$\int_{-\infty}^{\infty} S(\omega)\, d\omega$$

is also finite and so the function $S(\omega)$ is well-behaved in some sense. Note that Dirac deltas could exist in $S(\omega)$ without affecting this argument.

The upshot is that the power density can be defined for stochastic signals, but sequence of steps used to obtain the power density of a signal of a deterministic signal may not work for a stochastic signal.

Hence we must introduce another description - the autocorrelation function.

2.2 Correlation, autocorrelation

Up till now we have assumed independence of random variables. What if that is not true, and how to measure it?

One way is via a correlation function. Say we have two RVs, ξ, η . CF is:

$$C_{\xi\eta} = M\left[(\xi - M(\xi))(\eta - M(\eta))\right]$$

Rewriting:

$$C_{\xi\eta} = M[\xi\eta] - M(\xi)M(\eta)$$

For $\xi = \eta$, we have a familiar result:

$$C_{\xi\xi} = D^2(\xi) = M(\xi^2) - M(\xi)^2$$

We now define $\Gamma_{\xi\eta} = M(\xi\eta) = C_{\xi\eta} + M(\xi)M(\eta)$

[If ξ, η have DC components, we can incorporate them easily.]

Now consider the possibility of allowing time shifts between the two functions:

$$\Gamma_{\zeta\eta}(\tau) = M[\zeta(t)\eta(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\zeta(t)\eta(t+\tau)\,dt$$

This is the cross-correlation function.

We can apply this definition for a single stochastic process to get

$$\Gamma_{\zeta}(\tau) = M[\zeta(t)\zeta(t+\tau)]$$

or the autocorrelation function. [We require]

If there is no correlation between $\zeta(t)$, $\zeta(t+\tau)$ for any τ,
then $\Gamma_{\zeta}(\tau) \sim \delta(\tau)$.

If there is correlation, $\Gamma_{\zeta}(\tau)$ will tend to 0 for $\tau \to \infty$.

For stationary stochastic processes $\Gamma_{\zeta}(\tau)$ is independent of t . We are free to choose the integration interval $(-T, T)$.

The above equation is valid for stationary ergodic processes.

Properties:

1. (1)

$$\Gamma_{\zeta}(\tau) = \Gamma_{\zeta}(-\tau)$$

2. (2)

$$|\Gamma_{\zeta}(\tau)| \leq \Gamma_{\zeta}(0)$$

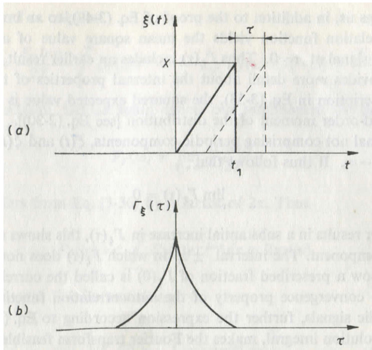


Figure 3-6 Asymmetrical stationary triangular wave. *(a)* Time function. *(b)* Autocorrelation function.

Figure 7: Signal and autocorrelation function.

Proof of (1): For your HW.

Proof of (2): $M[(\xi(t) \pm \xi(t+\tau))^2]$

$$= M[\underbrace{\xi(t)^2}_{\text{stationary}} \pm 2\xi(t)\xi(t+\tau) + \underline{\xi(t+\tau)^2}] \geq 0$$

$$M(\xi^2(t)) \geq M[|\xi(t)\xi(t+\tau)|]$$

$$\Gamma_\xi(0) \geq |\Gamma_\xi(\tau)| \quad \checkmark$$

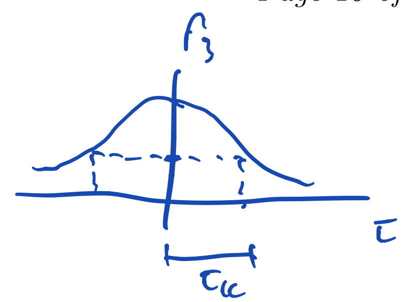
Related result: ACF yields the mean square value of a signal at $\tau = 0$. $\Gamma_\xi(\tau)$ thus provides additional information on the stochastic signal.

If a signal is not periodic, $\xi(t)$ & $\xi(t+\tau)$ are independent as $\tau \to \infty$. Thus

$$\lim_{\tau \to \infty} \Gamma_\xi(\tau) = 0$$

If τ increases and an increase in $\Gamma_\xi(\tau)$ is observed, a periodic component is present in the signal.

The interval $\pm\tau_k$ in which $\Gamma_\xi(\tau)$ does not decrease permanently below a given fraction of $\Gamma_\zeta(0)$ is called the correlation interval.



The convergent property of $\Gamma_\zeta(\tau)$ has great implications for the Fourier transform. Consider a signal limited to a time window as before. We have the ACF:

$$\Gamma_\zeta(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \zeta(t)\,\zeta(t+\tau)\,dt \rightarrow \underset{(-T,T)}{\frac{1}{2T}}\int_{-\infty}^{\infty} \zeta_T(t)\,\zeta_T(t+\tau)\,dt$$

Take a Fourier transform of $\Gamma_\zeta(\tau)$:

$$F_\Gamma(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}\left[\frac{1}{2T}\int_{-\infty}^{\infty} \zeta_T(t)\,\zeta_T(t+\tau)\,dt\right]d\tau$$

Add a factor of $1 = e^{j\omega t}\, e^{-j\omega t}$:

$$F_\Gamma(\omega) = \int_{-\infty}^{\infty} \frac{1}{2T}\int_{-\infty}^{\infty} \zeta_T(t)\, e^{j\omega t}\, \zeta_T(t+\tau)\, e^{-j\omega(t+\tau)}\,dt\,d\tau$$

We can do either integration first. Switch the order:

$$F_\Gamma(\omega) = \int_{-\infty}^{\infty} \frac{1}{2T}\left[\int_{-\infty}^{\infty} \zeta_T(t+\tau)\, e^{-j\omega(t+\tau)}\,d\tau\right] \zeta_T(t)\, e^{j\omega t}\,dt$$

Define $t+\tau = \theta$ so that $d\tau = d\theta$,

$$\int_{-\infty}^{\infty} \zeta_T(\theta)\, e^{-j\omega\theta}\,d\theta = F_T(\omega)$$

Note that $F_T(\omega)$ depends only on ω. $F_\Gamma(\omega)$ is now

$$F_\Gamma(\omega) = \frac{F_T(\omega)}{2T}\int_{-\infty}^{\infty} \zeta_T(t)\, e^{j\omega t}\,dt$$

$$= \frac{F_T(\omega)\,F_T(-\omega)}{2T} = \frac{|F_T(\omega)|^2}{2T}$$

Now let's remove the requirement of finite T. Define

$$F_r(\omega) = \lim_{T\to\infty} \frac{|F(\omega)|^2}{2T}$$

Recalling the earlier definition of power spectrum, we see that

$$F_r(\omega) = \int_{-\infty}^{\infty} r_3(\tau) e^{-j\omega\tau} d\tau = 2\pi\, s(\omega)$$

Then, if we inverse Fourier transform, we find:

$$r_3(\tau) = \int_{-\infty}^{\infty} s(\omega) e^{j\omega\tau} d\omega$$

In other words, the ACF $r_3(\tau)$ and power spectrum $s(\omega)$ are related by a Fourier transform. We have derived the Wiener-Khintchine theorem.

Common forms and notation:

Since $r_3(\tau)$ is a real, even function,

$$F_r(\omega) = \int_{-\infty}^{\infty} r_3(\tau) \cos(\omega\tau) d\tau = 2\int_0^{\infty} r_3(\tau)\cos(\omega\tau) d\tau$$

$$= 2\pi s(\omega)$$

$$F_r(\omega) = \begin{cases} \pi S'(\omega) & 0 < \omega < \infty \\ s(f) & -\infty < f < \infty \\ \frac{1}{2} S'(f) & 0 < f < \infty \end{cases}$$

Relation to autocorrelation function:

$$r_3(\tau) = \int_{-\infty}^{\infty} s(\omega) \cos\omega\tau\, d\omega = \int_0^{\infty} S'(\omega) \cos\omega\tau\, d\omega$$

$$= \int_{-\infty}^{\infty} s(f) \cos(2\pi f\tau)\, df = \int_0^{\infty} S'(f) \cos(2\pi f\tau)\, df$$

At $\tau = 0$: $\Gamma_\xi(0) = \int_{-\infty}^{\infty} S(\omega)\, d\omega = M[\xi^2(t)]$

as we have derived before.

Gaussian stochastic processes are those that have an ensemble density function following a multi-dimensional Gaussian at any fixed time instant. A nice property is that these distributions remain Gaussian even after a linear transformation.

More concretely: take two random variables, $\xi_1(t)$, $\xi_2(t+\tau)$. The values are x_1 and x_2. Let $M(x_1) = M(x_2) = 0$, variance σ_x^2 , and let them follow a two-variable normal distribution $f(x_1, x_2)$. Then

$$\sigma_x^2 = M[\xi^2(t)] = M[\xi^2(t+\tau)] = \int_{-\infty}^{\infty} x^2 f(x)dx \tag{1}$$

$$\Gamma_\xi(\tau) = M[\xi(t)\xi(t+\tau)] = \int\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2) dx_1 dx_2 \tag{2}$$

The process is Gaussian if:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_x^2\sqrt{1-\rho}} e^{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)/(2\sigma_x^2(1-\rho^2))} \tag{3}$$

$$\rho = \rho_x(\tau) = \frac{\Gamma_\xi(\tau)}{\sigma_x^2} \tag{4}$$

The common density function of instantaneous values always has a normal distribution, no matter the value of τ . ρ depends on τ .

3 Typical autocorrelation functions and their power spectra

3.1 Constant signal

Function: $u(t) = U = \text{const}, \quad -\infty < t < \infty$

Autocorrelation function:

$$\Gamma(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, u(t+\tau)\, dt = U^2$$

This value is also the mean square value. U^2

Power spectrum: $S(\omega) = \frac{1}{\pi} \int_0^{\infty} U^2 \cos(\omega\tau)\, d\tau = \frac{U^2}{\pi} \cdot \pi\, \delta(\omega)$

$$= U^2 \delta(\omega)$$

Reminder of Dirac delta properties:

$$\int_{-\varepsilon}^{\varepsilon} \delta(\omega)\, d\omega = 1 \quad \text{for any } \varepsilon > 0$$

$$\delta(-\omega) = \delta(\omega)$$

3.2 Sinusoidal signal

Signal: $u(t) = U \sin(\omega_0 t + \phi)$

Autocorrelation function:

$$\Gamma(\tau) = \lim_{T \to \infty} \frac{U^2}{2T} \int_{-T}^{T} \sin(\omega_0 t + \phi) \sin(\omega_0 t + \omega_0 \tau + \phi)\, dt$$

$$= \frac{U^2}{2} \cos(\omega_0 \tau)$$

Since $\Gamma(\tau)$ is independent of phase ϕ, this result is valid for $\cos$ as well.

Power spectrum:

$$S(\omega) = \frac{u^2}{2\pi}\int_0^\infty \cos\omega_0\tau \cos\omega\tau \, d\tau$$

$$= \frac{u^2}{2\pi}\left[\int_0^\infty \frac{\cos[(\omega-\omega_0)\tau]}{2} d\tau + \int_0^\infty \frac{\cos[(\omega+\omega_0)\tau]}{2} d\tau\right] = \frac{u^2}{2}\left(\frac{\delta(\omega-\omega_0)}{2} + \frac{\delta(\omega+\omega_0)}{2}\right)$$

It consists of two weighted Dirac deltas symmetric about the origin.

Single-sided power spectrum: $\mathcal{S} = \frac{u^2}{2}\delta(\omega-\omega_0) \quad (0 < \omega < \infty)$

As expected, all spectral power is in one spectral line for this sinusoid.

If we know the time function, getting spectral power is straightforward. For stochastic signals, we generally can measure $S(\omega)$. We can then get $\Gamma(\tau)$ but not the original time signal since the phase information is lost.

Consider band-limited white noise with a single-sided power spectrum as:

$$\mathcal{S}(\omega) = \begin{cases} A & \omega_0 - \frac{\Delta\omega}{2} < \omega < \omega_0 + \frac{\Delta\omega}{2} \\ 0 & \text{o/w} \end{cases}$$

Autocorrelation function:

$$\Gamma_3(\tau) = \int_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\Delta\omega/2} A\cos\omega\tau \, d\omega \simeq A\,\Delta\omega\cos\omega_0\tau \frac{\sin\tau\Delta\omega/2}{\tau\Delta\omega/2}$$

Let's consider two cases.

1. $\tau\Delta\omega \ll 1$ so that $\Gamma_3(\tau) \simeq A\,\Delta\omega\cos\omega_0\tau$

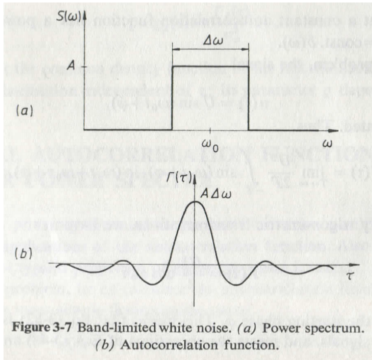


Figure 3-7 Band-limited white noise. *(a)* Power spectrum. *(b)* Autocorrelation function.

Figure 8: Band-limited white noise and ACF.

So, in the limit of low bandwidths and short correlation times, a stochastic signal can be approximated by a sinusoidal signal of frequency ω_0. Now, previously we found from the Rayleigh distribution that narrow-band noise can be thought of as a carrier with amplitude and phase stochastically modulated.

The above analysis tells us that actually the carrier amplitude can be approximated as constant (in this limit).

$\tau \Delta\omega \ll 1$

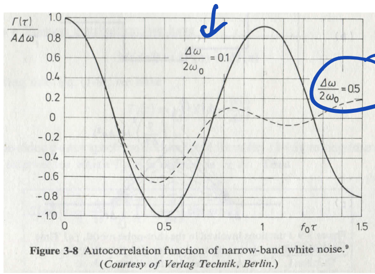


Figure 3-8 Autocorrelation function of narrow-band white noise.[9]
(Courtesy of Verlag Technik, Berlin.)

Figure 9: ACF of narrow-band white noise.

General rule: smaller the bandwidth, the more deterministic the signal is ↔ the slower the random phase changes. We will use an approximation based on this later.

2. Say $\frac{\Delta\omega}{2} = \omega_0$ (wideband). Then

$$\Gamma_3(\tau) = \int_0^{\Delta\omega} A \cos\omega\tau \, d\omega = A\Delta\omega \frac{\sin \tau\Delta\omega}{\tau\Delta\omega}$$

As $\Delta\omega \uparrow$, $|\Gamma_3(\tau)|$ goes to zero faster. relative to NB case. In the limit, we have:

$$\lim_{\Delta\omega \to \infty} \Gamma_3(\tau) = A\pi\delta(\tau)$$

Therefore, the autocorrelation time interval of ideal white noise with infinite bandwidth is zero → no correlation between instantaneous values.

3.3 Shot noise

Let's return to our early calculation of shot noise power spectrum.

Here is a sketch of the current fluctuation at time t and $t-\tau$:

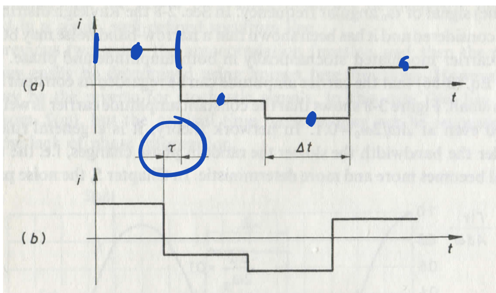
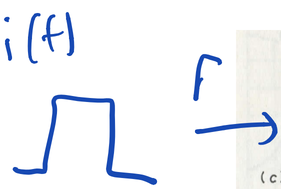


Figure 10: Shot noise current versus time.

In equations: $$\underbrace{i(t)}_{\text{fluctuation}} = \frac{q}{\Delta t}\left[n(t) - \bar{n}\right] = \frac{q}{\Delta t}\Delta n(t)$$

The autocorrelation function is:



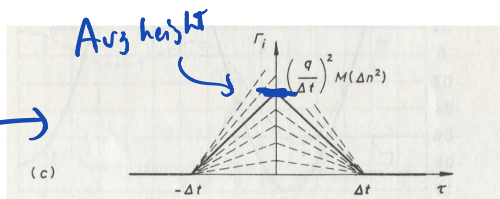


Figure 11: ACF of shot noise current.

To compute this function, $i(t)$ has to be shifted left or right and the product of instantaneous values has to be averaged for a long time.

Now, the value $\Delta n(t), \Delta n(t+\Delta t), \ldots$, are independent -> the autocorrelation interval is confined to $\pm \Delta t$.

The autocorrelation function of rectangular pulses has a triangle shape.

To get the average height, get $M(\Delta n^2)$ by setting $\tau = 0$ in ACF.

Average height is then $$\left(\frac{q}{\Delta t}\right)^2 M(\Delta n^2)$$

So that $\Gamma_i(\tau) = \begin{cases} \left(\frac{q}{\Delta t}\right)^2 M(\Delta n^2)\left(1 - \frac{|\tau|}{\Delta t}\right) & \text{if } |\tau| \leq \Delta t \\ 0 & \text{o/w} \end{cases}$

We already figured out that n has a Poisson distribution. Therefore,

$$M(\Delta n^2) = D^2(n) = M(n)$$

So we get: $\Gamma_i(\tau) = \frac{q}{\Delta t} I \left(1 - \frac{|\tau|}{\Delta t}\right) \quad \left[I = \frac{q\, M(n)}{\Delta t} \right]$

We get the single-sided power spectrum in the usual way (remembering that Γ_i is defined within only $\pm \Delta t$).

$$S'(\omega) = \frac{2}{\pi} \frac{q}{\Delta t} I \int_0^{\Delta t} \left(1 - \frac{\tau}{\Delta t}\right) \cos\omega\tau \, d\tau = \frac{1}{\pi} q I \frac{\sin^2(\omega\Delta t/2)}{(\omega\Delta t/2)^2}$$

If $\omega\Delta t \ll 2\pi$, sinc -> 1, and

$$S(\omega) = \frac{1}{\pi} q I$$

Transforming to frequency rather than angular frequency, we get our original result:

$$S'(f) = 2\pi S'(\omega) = 2qI \quad \checkmark$$

The mean square value of the fluctuation in a frequency band Δf is

$$M[i^2(t)]_{\Delta f} = \int_f^{f+\Delta f} S(f)\, df = 2qI\Delta f$$

This calculation helps us see a limitation on Δt : it cannot be too small because otherwise the assumption of independence of current in each time interval is not true.

In the shot noise derivation, we subdivided time into ~~arbitrary~~ intervals and found current pulses due to this subdivision.

Some noise sources naturally have a pulse waveform (e.g. burst noise and processes related to voltage breakdown). A time function of "random telegraph noise" is

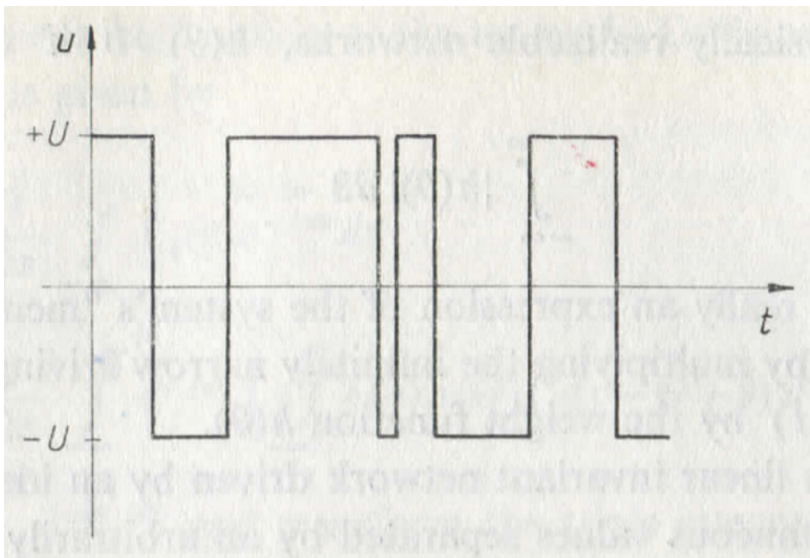


Figure 3-10 Random telegraph signal.

Figure 12: Telegraph noise.

Lorentzian

You will see how to obtain the spectral power of this signal in the HW.

4 Response of a linear network driven by a stochastic signal

For small signals, we describe electronic devices as linear networks. In a transistor, several different noise sources can simultaneously exist. In between them can exist resistive and reactive network elements. The ACF and power spectrum that is measured is affected by this network.

The most general relation between the input $x(t)$ and output $y(t)$ of a linear network with weight function $h(\theta)$ is

$$y(t) = \int_{-\infty}^{\infty} h(\theta)\, x(t-\theta)\, d\theta$$

By causality, physical networks have $h(\theta) = 0,\ \theta < 0$.

So that the output does not diverge, we require

Now consider an ideal white noise source with independent instantaneous values for an arbitrarily short time interval ( ). By the linear relation, the output will contain the driving function value at a time instant, as well as earlier input values owing to the 'memory' of . Therefore, the ACF of the driving and response waveforms are not the same in general.

Let the input driving signal be stationary and ergodic with ACF .

We want the ACF of the response .

To get it, first compute :

Recall the definition of ACF:

So we need to integrate in time. We can exchange the order of integrations:

Let's use Fourier transforms to simplify this expression. Recall that

So the Fourier transform of ACF is:

Multiply by , and exchange order of integrals again:

Since is real (it is the impulse response of the electrical network consisting of e.g. voltage measurements from a scope), the first two terms are a conjugate pair:

So in the frequency domain, we get the output response simply by multiplication.

We often use this equation rather than the time-domain version. Relations:

Example: ACF of the output signal of a single-pole low pass filter. Transfer function:

Input = , single-sided power spectrum:

Filter output:

ACF:

Schematic figures:

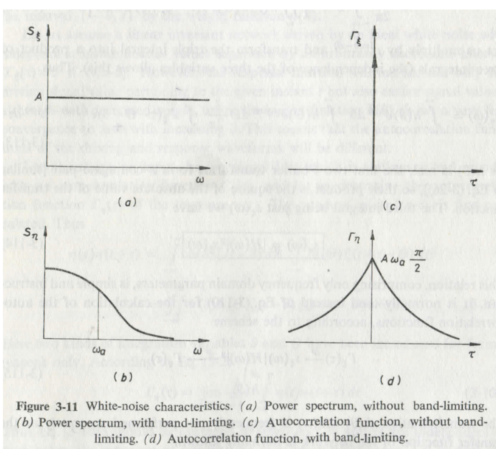


Figure 13: Output through a band-limiting network.

If the input signal has a normal distribution, output signal will also. In fact, a band-limited network output will be closer to normal because output is a linear combination of instantaneous values weighted by . So, the sum distribution should approach the normal distribution due to central limit theorem.

In practice, we need the time constant of the network to be larger than the AC interval so that the samples are independent.