Microwave Noise in Semiconductor Devices

Module 5

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1 Markovian processes and Langevin dynamics

So far we have considered noise sources in equilibrium (Johnson thermal noise) and non-equilibrium but in the absence of scattering (shot noise).

However, in a real device electrons are heated by a field and undergo scattering. As mentioned in the last module, this noise source corresponds to a physical temperature of thousands of K and hence is quite important.

In this module we want to build up the mathematical description of fluctuations in non-equilibrium gases (taking non-interacting electrons in a solid as our gas of interest).

To do that we have to go back over a few mathematical topics in more detail.

2 More on random and Markovian processes

[Kogan Chap 1]

2.1 Correlation functions and conditional probability

First let us revisit the correlation function and include some additional elements of probability theory.

Take as a random variable with mean value . The fluctuation, is also random.

The correlation function is a non-random characteristic of the kinetics of the random fluctuations. It is a useful way to understand how fluctuations evolve on average.

As we discussed before, the correlation function is defined as the ensemble average of the fluctuation of system at two time instants, :

$$\psi_x(t_1, t_2) \equiv \langle \delta x(t_1)\delta x(t_2)\rangle = \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle \langle x(t_2)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta x_i(t_1)\delta x_i(t_2) \quad (1)$$

For a stationary system this correlation function depends only on .

Assuming an ergodic system, we can also express the ensemble average as a time average as:

$$\psi_x(t_1 - t_2) \equiv \overline{\delta x(t_1)\delta x(t_2)} = \lim_{t_m\to\infty} \frac{1}{t_m}\int_{-t_m/2}^{t_m/2} dt\delta x(t_1 + t)\delta x(t_2 + t) \quad (2)$$

At the correlation function is the variance ( ).

As the correlation function goes to . To understand that, consider a fluctuation in the system of the ensemble at instant , call it . For but with close to , the fluctuation of each system in ensemble has no time to change its value substantially, so likely has the same sign as . Therefore for most systems in ensemble. As increases, more systems have sign of opposite to . At large enough , positive and negative fluctuations are equally probably and hence .

Now some new ideas. Previously we discussed density and distribution functions.

Recall that the distribution function is

or the probability that the random quantity at instant is less than .

We can generalize them to multiple dimensions as follows:

$$W_2(x_1, t_1; x_2, t_2) = P(x(t_1) \leq x_1; x(t_2) \leq x_2) \tag{3}$$

$$W_n(x_1, t_1; ...; x_n, t_n) = P(x(t_1) \leq x_1; ...; x(t_n) \leq x_n) \tag{4}$$

, for instance, is the probability that random quantity at instant is less than and at instant is less than .

As before, we can introduce single and multivariable density functions:

$$w_1(x_1, t_1) = \frac{\partial W_1(x_1, t_1)}{\partial x_1} \tag{5}$$

$$w_2(x_1, t_1; x_2, t_2) = \frac{\partial^2 W_2(x_1, t_1; x_2, t_2)}{\partial x_1 \partial x_2} \tag{6}$$

and so on.

As an example, the 2D probability density function

$$w_2(x_1, t_1; x_2, t_2)dx_1dx_2 = P(x_1 \leq x(t_1) \leq x_1 + dx_1; x_2 \leq x(t_2) \leq x_2 + dx_2) \tag{7}$$

gives the probability of random var at instant to have a value between , and also at instant to have value between .

The probability density functions have to be self-consistent in that lower order functions ( ) can be obtained from higher order ones ( ) by integrating out extra variables

$$w_k(x_1, t_1; ...; x_k, t_k) = \int dx_{k+1} dx_n w_n(x_1, t_1; ...; x_n, t_n) \tag{8}$$

Recall that random processes are stationary if all distributions are invariant under a shift of all time points . That implies that does not depend on , depends only on , and so on.

Using the density functions we define mean as usual:

$$\langle x(t)\rangle = \int x w_1(x,t)dx \tag{9}$$

The order central moment is the average value of the fluctuation

$$\langle(\delta x(t))^r\rangle = \int (\delta x)^r w_1(x,t)dx \tag{10}$$

We can also compute correlation functions:

$$\langle\delta x(t_1)...\delta x(t_n)\rangle = \int dx_1...dx_n \delta x_1...\delta x_n w_n(x_1,t_1;...;x_n,t_n) \tag{11}$$

Therefore, we have a new way to think to think about correlation functions in terms of the density functions. This point of view is closely related to the ensemble average rather than time average.

With this definition, we can use elements of probability theory to get more insight. Let's define the two-time correlation function as:

$$\psi_x(t_1,t_2) = \int dx_1 dx_2 \delta x_1 \delta x_2 w_2(x_1,t_1;x_2,t_2) \tag{12}$$

You may recall Bayes theorem, which relates joint and conditional probabilities. Applying it here, we see that

$$w_2(x_1,t_1;x_2,t_2) = w_1(x_1,t_1)P(x_2,t_2|x_1,t_1) \tag{13}$$

We have defined , the 1D density function of at instant , and the conditional probability , which is the probability for random variable at time to have a value in the interval , given that in a previous instant its value was .

Some helpful notation:

$$\langle \delta x(t_2) | \delta x(t_1) \rangle = \int dx_2 \delta x_2 P(x_2, t_2 | x_1, t_1) \tag{14}$$

which is a conditional mean: it is the mean value of fluctuation at instant with the condition that at a previous instant the value was .
Using this for the correlation function, we have

$$\psi_x(t_1, t_2) = \int dx_1 \delta x_1 w_1(x_1, t_1) \langle \delta x(t_2) | \delta x(t_1) \rangle \tag{15}$$

Here describes the average evolution of the fluctuation with an initial value of at time .
Note that the absolute value of a fluctuation for a given starting value can increase or decrease with time. On average it usually decreases following our intuition that fluctuations are damped in stable systems.
We can now think of the correlation function as the value averaged over initial values .

Let's get some intuition for how the conditional probability behaves.
Consider a subset of the ensemble for which fluctuations at instant are close to . As time evolves, the fluctuations of these systems will become randomized, different from each other, and have a distribution the same as that of the ensemble.
But, the mean value of fluctuation of the ensemble is . So the conditional average value of the fluctuation tends to :

$$\lim_{(t_2 - t_1) \to \infty} \langle \delta x(t_2) | \delta x(t_1) \rangle = 0 \tag{16}$$

Another case: consider a Gaussian random process and its conditional mean fluctuation and its correlation function.

Recall that for a 2D Gaussian process we have

$$w_2(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp\left[-\frac{1}{2(1-\rho_{12}^2)}\left[\frac{(\delta x_1)^2}{\sigma_1^2} + \frac{(\delta x_2)^2}{\sigma_2^2} - 2\rho_{12}\frac{\delta x_1}{\sigma_1}\frac{\delta x_2}{\sigma_2}\right]\right] \tag{17}$$

where $\sigma_i^2 = \langle(\delta x_i)^2\rangle$ and $\rho_{12} = \langle\delta x_1 \delta x_2\rangle / \sigma_1\sigma_2$. Using this formula for the conditional probability

$$P(\delta x_2, t_2; \delta x_1, t_1) = \frac{w_2}{w_1} \tag{18}$$

and integrating out , we get

$$\frac{\langle\delta x(t)|\delta x_0, t = 0\rangle}{\delta x_0} = \frac{\psi_x(t)}{\langle(\delta x)^2\rangle} \tag{19}$$

Therefore for a Gaussian process, the time-dependence of the conditional mean fluctuation is the same as for the correlation function for any initial fluctuation .

Example: if is monotonically decreasing with , the fluctuation at on average decreases at any (arbitrarily small) .
[Remember this behavior occurs on average, not for instantaneous values.]
We can define a correlation matrix between several random quantities:

$$\psi_{\alpha\beta}(t_1, t_2) = \langle\delta x_\alpha(t_1)\delta x_\beta(t_2)\rangle \tag{20}$$

If we have an autocorrelation, otherwise we have cross-correlation.

Until now we have considered classical processes. In quantum mechanics observable quantities are represented as operators, and the observable must be Hermitian (since the observable is a real, physical measurable quantity). However, our definition of correlation function is not Hermitian!

To solve that problem, define the observable for the correlation function as

$$\frac{1}{2}\{\hat{x}(t_1), \hat{x}(t_2)\} \equiv \frac{1}{2}(\hat{x}(t_1)\hat{x}(t_2) + \hat{x}(t_2)\hat{x}(t_1)) \tag{21}$$

Now in classical systems we compute an average. In quantum systems we need to take an expectation value over the density matrix of the quantum mechanical system.

The expression for the correlation function is then

$$\psi_x(t_1, t_2) = \frac{1}{2}Tr(\rho\{\hat{x}(t_1), \hat{x}(t_2)\}) \equiv \frac{1}{2}\left\langle\{\hat{x}(t_1), \hat{x}(t_2)\}\right\rangle \tag{22}$$

By definition, it is symmetric with respect to .

For stationary systems, must therefore be even.

$$\psi(t_1, t_2) = \psi(t_2, t_1) \rightarrow \psi(t_1 - t_2) = \psi(t_2 - t_1) \tag{23}$$

For many random variables, we have a correlation matrix

$$\psi_{\alpha\beta}(t_1, t_2) = \frac{1}{2}\left\langle\{\hat{x}_\alpha(t_1), \hat{x}_\beta(t_2)\}\right\rangle \tag{24}$$

The matrix elements satisfy

$$\psi_{\alpha\beta}(t_1, t_2) = \psi_{\beta\alpha}(t_2, t_1) \tag{25}$$

and for a stationary system,

$$\psi_{\alpha\beta}(t_1 - t_2) = \psi_{\beta\alpha}(t_2 - t_1) \tag{26}$$

There are further relationships between these matrix elements due to time reversal symmetry. The equations of motion of classical or quantum particles are symmetric if $t \rightarrow -t$ and

$\vec{B} \to -\vec{B}$. That means

$$\psi_{\alpha\beta}(t_1 - t_2; \vec{B}) = \pm\psi_{\alpha\beta}(t_2 - t_1; -\vec{B}) \tag{27}$$

The + sign occurs if $x_\alpha(t)$ and $x_\beta(t)$ are both invariant or both change their sign under time reversal.

Example: velocity changes sign under time inversion.

The $-$ sign occurs if only one quantity changes sign.

If $\vec{B} = 0$, we see $\psi_{\alpha\beta}$ must be a strictly even or odd function of $t_1 - t_2$.

We get more properties ($\vec{B} = 0$) at short times $t_1 - t_2 \equiv t$.

For a + sign, we have

$$\begin{aligned} \psi_{\alpha\beta}(t) &= \psi_{\alpha\beta}(-t) && (28) \\ \psi_{\alpha\beta}(0) + \dot{\psi}_{\alpha\beta}|_0 &= \psi_{\alpha\beta}(0) - \dot{\psi}_{\alpha\beta}|_0 && (29) \end{aligned}$$

so that $\dot{\psi}_{\alpha\beta}|_0 = 0$.

For a $-$ sign, we have

$$\lim_{t\to 0} [\psi_{\alpha\beta}(t) = -\psi_{\alpha\beta}(-t)] \tag{30}$$

so that $\psi_{\alpha\beta}(0) = 0$.

As we have done before, we can take a Fourier transform of the correlation function of a stationary system:

$$\psi_{\alpha\beta}(\omega) = \int_{-\infty}^{\infty} d(t_1 - t_2) e^{j\omega(t_1 - t_2)} \psi_{\alpha\beta}(t_1 - t_2) \tag{31}$$

We previously derived that $\psi_{\alpha\beta}(\omega) = \psi^*_{\alpha\beta}(-\omega)$ since correlation functions are real.

We also have:

$$\begin{aligned} \psi_{\alpha\beta}(\omega) &= \psi_{\beta\alpha}(-\omega) = \psi^*_{\beta\alpha}(\omega) && (32) \\ \psi_{\alpha\beta}(\omega, \vec{B}) &= \pm\psi_{\alpha\beta}(-\omega, -\vec{B}) = \pm\psi^*_{\alpha\beta}(\omega, -\vec{B}) && (33) \end{aligned}$$

So we see $\psi_{\alpha\beta}(\omega)$ is a Hermitian matrix.

The second equation means that if $\vec{B} = 0$ then $\psi_{\alpha\beta}(\omega)$ are purely real or purely imaginary.

2.2 Spectral density of noise

Let's revisit the spectral density of noise in this new notation. The general setup is as follows. We have a source of time-dependent fluctuations $\delta x(t)$. They are passed through a bandpass filter. We read the power of the filtered signal.

Taking the signal to be measured in a long time interval $t_m \to \infty$, we represent the random signal in a Fourier series:

$$\delta x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta x(\omega) e^{-j\omega t} \tag{34}$$

[Remember that this is a bit loose notation since $\delta x(\omega)$ may not formally exist.]

Now $\delta x(t)$ is real, so $x(-\omega) = x^*(-\omega)$. Then

$$\delta x(t) = \int_{0}^{\infty} \frac{d\omega}{2\pi} \left[\delta x(\omega) e^{-j\omega t} + \delta x^*(\omega) e^{+j\omega t}\right] \tag{35}$$

The filtered signal is

$$\delta x(t; \bar{f}, \Delta f) = \int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2} \frac{d\omega}{2\pi} \left[\delta x(\omega) e^{-j\omega t} + \delta x^*(\omega) e^{+j\omega t}\right] \tag{36}$$

The signal squared = noise power. It depends on time and random fluctuates about the mean value $[\delta x(t;\bar{f},\Delta f)]^2$. Perform an ensemble average to get

$$\langle \delta x(t;\bar{f},\Delta f)^2\rangle = \int\int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2} \frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\left\langle\left[\delta x(\omega')e^{-j\omega' t}+\delta x^*(\omega')e^{+j\omega' t}\right]\left[\delta x(\omega'')e^{-j\omega'' t}+\delta x^*(\omega'')e^{+j\omega'' t}\right]\right\rangle \tag{37}$$

Now we need to figure out what the ensemble average is. Go back to our earlier discussion of correlation function. Using the Fourier series representation we write

$$\psi_x(t_1-t_2) = \langle \delta x(t_1)\delta x(t_2)\rangle = \int\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\langle \delta x(\omega')\delta x(\omega'')\rangle\, e^{-j\omega' t_1 - j\omega'' t_2} \tag{38}$$

We assumed a stationary random process, so ψ_x can only depend on t_1-t_2 . Therefore, the ensemble average $\langle\rangle$ must be nonzero only for $\omega'+\omega''=0$. E.g.

$$\langle\rangle \propto 2\pi\delta(\omega'+\omega'') \tag{39}$$

We get the coefficient by realizing the equation is now the inverse Fourier transform of the correlation function $\psi_x(t_1-t_2)$.

For that to happen we need

$$\langle \delta x(\omega')\delta x(\omega'')\rangle = 2\pi\psi_x(\omega')\delta(\omega'+\omega'') \tag{40}$$

Putting it all together, we get

$$\overline{\delta x(t;\bar{f},\Delta f)^2} = 2\int_{\bar{f}-\Delta f/2}^{\bar{f}+\Delta f/2} df\,\psi_x(\omega) \approx S_x(\bar{f})\Delta f \tag{41}$$

We again find that the power transmitted through the filter is proportional to Δf , and we have defined the spectral density of noise per unit bandwidth, $S_x(\bar{f})$. The definition per the Wiener-Khintchine theorem is

$$S_x(f) = 2\int_{-\infty}^{\infty} d(t_1-t_2)e^{j\omega(t_1-t_2)}\psi_x(t_1-t_2) \equiv 2\psi_x(\omega) \tag{42}$$

We can also show that:

$$S_x(f) = 2 \lim_{t_m \to \infty} \frac{1}{t_m} \left| \int_{-t_m/2}^{t_m/2} dt e^{j\omega t} \delta x(t) \right|^2 \tag{43}$$

If we have multiple fluctuating quantities we get a spectral density matrix. Say we have two fluctuating quantities, $\delta x_\alpha(t)$ and $\delta x_\beta(t)$. Then

$$S_{\alpha\beta}(f) = 2 \lim_{t_m \to \infty} \frac{1}{t_m} \int_{-t_m/2}^{t_m/2} dt_1 e^{j\omega t_1} \delta x_\alpha(t_1) \int_{-t_m/2}^{t_m/2} dt_2 e^{j\omega t_2} \delta x_\alpha(t_2) = 2\psi_{\alpha\beta}(\omega) \tag{44}$$

The spectral density matrix is just 2× the the Fourier transform of the correlation matrix and thus has the same properties we derived before.
Using various definitions of Fourier transforms, we find that the mean squared signal equals the variance of fluctuations:

$$\langle (\delta x)^2 \rangle = \psi_x(t_1 - t_2) = \int_0^\infty df S_x(f) \tag{45}$$

Another way to think about it is time-domain and frequency-domain relationships: the ACF and spectral density are related by Fourier transforms. Therefore integrating the spectral density over all frequencies yields the ACF at zero time which is just the variance of the signal, and integrating the ACF over all time gives the spectral density at zero frequency.

3 Markov processes

Hopefully the above was a good review. Now let's discuss some new concepts in probability. To handle non-equilibrium noise we need to be able to describe random processing evolving in time. The simplest type of process to analyze is a Markov process.

A Markov process is defined as one in which the probability of a process occurring at a given time instant depends only on the state of the system at that instant.

Here is some intuition for why that is often a good approximation.

Example 1: consider temperature fluctuations in a macroscopic body that exchanges heat with a reservoir. A fluctuation in temperature involves changes in a large number of microscopic parameters (since temperature is defined as occupation over all energy states). On the timescale of a measurement, vast numbers of microscopic processes occur to adjust to energy fluctuations that define temperature. Given the large number of microscopic processes that occur between each measurement, it seems reasonable to assume that the system loses memory of any other information on its states prior to the current state.

More precisely, the conditional probability of the temperature fluctuation being at time instant , if at the preceding measurement at it was , depends only on this measurement and not any other previous ones.

Example 2: Mobile impurities in solids move by hopping over an energy barrier. The rate of hop attempts, , is often much larger than the actual hop rate. After hopping, the impurity attempts to hop many times again, in the process equilibrating with the state of its present location. It is again reasonable to assume that in this process it loses memory of how it ended up at its current site.

Now let's put some math in. Say we have measured a random signal at several instants, . Take the time intervals to be many times greater than the characteristic microscopic time.

We discussed that the random process can be characterized by a density function:

Which gives the probability that has value in at time , value at time , and so on.

Use of Bayes theorem tells us that

Here is the conditional probability that is in given that at preceding instants it took the values .

So in general, the conditional probability depends on the entire previous trajectory.

Now let's use the Markovian assumption: the conditional probability only depends on the value at instant that precedes :

In short, we take the system to possess no long-term memory. The conditional probability function is called transition probability.

A homogeneous Markov process depends only on . It may not necessarily be stationary, but all stationary processes are homogeneous. We will only consider stationary processes.

This definition can be extended to random quantities by collecting them in a vector and making the transition probability a matrix .

The transition probability must satisfy a very restrictive condition. Say is any intermediate time between , so that .

The probability of is a sum of transitions through all intermediate :

This condition is called the Smoluchowski equation. It is very restrictive and helps constrain the possible transition probability functions that are physical.

Other conditions.

The probability to transition to all other states is unity:

If is the stationary density function, when we average the transition probability over all initial values with density function , we must get

If is small compared with the characteristic time, has no time to deviate from its initial value. Therefore,

If is much longer than the relaxation time, the system loses memory of its initial condition and the transition probability is just the density at :

Because of this property, we often use (little) defined as:

It goes to zero as . The relations for this function are:

The Smoluchowski equation remains unchanged.

From this equation, we can also derive a differential equation that governs the kinetics of transition probabilities. They are referred to as the Kolmogorov equations.

Write the equation for maximum time and intermediate time :

Considering the properties we just derived, we can write:

Put this into the Smoluchowski equation:

Observe a finite difference for time derivative and take :

This is the Kolmogorov equation.

The quantities are the derivatives of wrt first time argument. If , we can loosely interpret it as the probability of transition per unit time from to .

From the properties of $P(x,t|x_0,t_0)$, we have

$$\int dx\lambda(x,x') = 0 \tag{46}$$

$$\int dx'\lambda(x,x')w(x') = 0 \tag{47}$$

We now want to derive equations for the kinetics of the correlation function of fluctuations, $\psi_{\alpha\beta}(t_1-t_2)$ which is defined as $\psi_{\alpha\beta} = \langle \delta x_\alpha(t_1)\delta x_\beta(t_2)\rangle$. They should depend on the $\psi_{\alpha\beta}(0)$, or the variances.

To start, we need to define how a small perturbation at some time instant relaxes. Say at instant t_0, $x(t_0) = x_0$. The conditional mean value of x_α at instant $t > t_0$ is

$$\langle x_\alpha(t)|x_0,t_0\rangle = \int dx x_\alpha P(x,t|x_0,t_0) \tag{48}$$

Recalling that $\langle x_\alpha\rangle = \int dx x_\alpha P(x,t|x_0,t_0)$, we can write the mean deviation of x_α from its mean value as

$$\langle \Delta x_\alpha(t)|x_0,t_0)\rangle = \int dx\Delta x_\alpha P(x,t|x_0,t_0) = \int dx x_\alpha p(x,t|x_0,t_0) \tag{49}$$

From the Kolmogorov equation, the time derivative of lhs yields

$$\frac{d}{dt}\langle \Delta x_\alpha(t)|x_0,t_0)\rangle = -\int dx\int dx'\Delta x_\alpha\lambda(x,x')p(x',t|x_0,t_0) \tag{50}$$

Taking the deviations $\Delta x'$ from mean $\langle x'\rangle$ as small, we write the integral term as

$$\int dx\Delta x_\alpha\lambda(x,x') = \Lambda_{\alpha\beta}\Delta x'_\beta \tag{51}$$

Now $\Lambda_{\alpha\beta}$ is a matrix with eigenvalues that give the inverse relaxation times of the system.

[Summation over repeated indices implied]. We therefore find

$$\frac{d}{dt}\left\langle \Delta x_\alpha(t)|x_0,t_0)\right\rangle = -\Lambda_{\alpha\beta}\left\langle \Delta x_\beta(t)|x_0,t_0)\right\rangle \tag{52}$$

With this equation for the conditional value of the fluctuation, we can find the equation for the correlation function. Rewrite $\psi_{\alpha\beta}(t_1-t_2)$ into 'one-sided' parts:

$$\psi_{\alpha\beta}(t_1-t_2) = \psi^+_{\alpha\beta}(t_1-t_2)+\psi^-_{\alpha\beta}(t_1-t_2) \tag{53}$$

$$\psi^+_{\alpha\beta}(t_1-t_2) = \Theta(t_1-t_2)\left\langle \delta x_\alpha(t_1)\delta x_\beta(t_2)\right\rangle \tag{54}$$

$$\psi^-_{\alpha\beta}(t_1-t_2) = \Theta(t_2-t_1)\left\langle \delta x_\beta(t_2)\delta x_\alpha(t_1)\right\rangle \tag{55}$$

From the previous definition of correlation function in terms of condition probabilities:

$$\psi^+_{\alpha\beta}(t) = \Theta(t)\int dxdx' x_\alpha P(x,t|x',0)x'_\beta w(x') \tag{56}$$

$$= \Theta(t)\int dxdx'\delta x_\alpha p(x,t|x',0)\delta x'_\beta w(x') \tag{57}$$

$$= \Theta(t)\int dx'\left\langle \delta x_\alpha(t)|x_0,0\right\rangle \delta x'_\beta w(x') \tag{58}$$

and $\psi^-_{\alpha\beta}(t)=\psi^+_{\beta\alpha}(|t|)$.

Now take a time derivative (taking $t=t_1-t_2$).

First, derivative of step function is delta function, and the coefficient is $\psi_{\alpha\beta}(0)$.

Second, the derivative of $p(x_1,t_1|x_2,t_2)$ is given by Kolmogorov equation.

Third, we use the approximation with the $\Lambda_{\alpha\beta}$ matrix.

Then we get

$$\frac{\partial\psi^+_{\alpha\beta}}{\partial t}+\Lambda_{\alpha\gamma}\psi^+_{\gamma\beta}=\psi_{\alpha\beta}(0)\delta(t) \tag{59}$$

Assume that we can diagonalize $\Lambda_{\alpha\beta}$. We can then solve this equation.

Call λ_m the eigenvalues and $\chi^{(m)}_\alpha$ the eigenvectors of $\Lambda_{\alpha\beta}$, of size M. Call $\phi^{(m)}_\alpha$ the eigenvector of the conjugate matrix $\Lambda^{\dagger}_{\alpha\beta}$ giving eigenvalue λ^*_m. By definition,

$$\Lambda_{\alpha\beta}\chi^{(m)}_{\beta} = \lambda_m \chi^{(m)}_{\alpha} \tag{60}$$

$$\Lambda^{\dagger}_{\alpha\beta}\phi^{(m)}_{\beta} = \lambda^*_m \phi^{(m)}_{\alpha} \tag{61}$$

Assume all eigenvalues are different. Normalized eigenfunctions are orthonormal:

$$\sum_{\alpha} \chi^{(m)}_{\alpha}\phi^{(n)*}_{\alpha} = \delta_{nm} \tag{62}$$

With these eigenvectors, we find that linear combinations of δx_α (weights given by values in eigenvectors) have a single exponential time constant λ_m^{-1} :

$$\Phi^+_{mn}(t) = \sum_{\alpha\beta} \phi^{(m)*}_{\alpha}\psi^+_{\alpha\beta}(t)\chi^{(m)}_{\beta} \tag{63}$$

Transforming to the eigenbasis, we get

$$\frac{\partial \Phi^+_{mn}}{\partial t} + \lambda_m \Phi^+_{mn} = \Phi^+_{mn}(0)\delta(t) \tag{64}$$

with the standard solution

$$\Phi^+_{mn}(t) = \Theta(t)\Phi^+_{mn}(0)e^{-\lambda_m t} \tag{65}$$

Therefore the original correlation function in terms of the eigenfunctions is:

$$\psi^+_{\alpha\beta}(t) = \Theta(t)\sum_{m\gamma} e^{-\lambda_m t}\chi^{(m)}_{\alpha}\phi^{(m)*}_{\gamma}\psi_{\alpha\beta}(0) \tag{66}$$

The matrix of spectral densities is obtained by Fourier transform:

$$S_{\alpha\beta}(f) = 2\sum_{m\gamma}\left[\frac{\chi^{(m)}_{\alpha}\psi_{\gamma\beta}(0)\phi^{(m)*}_{\gamma}}{\lambda_m - j\omega} + \frac{\chi^{(m)}_{\beta}\psi_{\gamma\alpha}(0)\phi^{(m)*}_{\gamma}}{\lambda_m + j\omega}\right] \tag{67}$$

Now consider that a single relaxation time exists only. Then we get a simple Lorentzian for the spectral density of noise:

That is why so many physical processes have a Lorentzian spectral density.

One discrepancy to discuss: note that the correlation function is generally a sum of exponentials that depend on . The time derivatives on either side of are clearly different and not equal to zero. That contradicts a property we derived before.

The resolution is that for Markovian processes we are implicitly assuming that the time interval in a Markov process is much bigger than the characteristic microscopic time. Therefore, the derived correlation functions do not apply for too small a time. Here is a plot that illustrates that point:

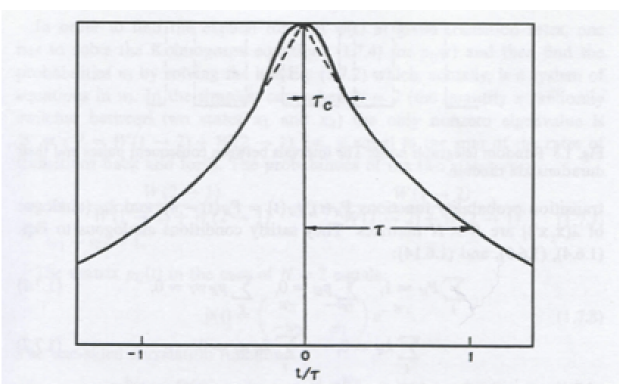

Fig. 1.4 Correlation function $\psi(t)$ of a random process with a single relaxation time τ. Solid curve: real decay of the correlation function. Dashed curve: purely exponential correlation function $\exp(-|t|/\tau)$ obtained in the Markov approximation. It deviates from the real one at small times $|t| < \tau_c$ which in Markov approximation are 'microscopic' times. From Lax (1960).

Figure 1: Markovian assumption.

4 Langevin approach

Here is another way to analyze random processes, originally described by Langevin in 1908. In the Markov approach, we got equations whose solutions were definite (not random) correlation functions or transition probabilities that characterize the random process.

In the Langevin approach, we solve for the actual random variables or fluctuations, then compute correlation functions by averaging in an appropriate way, e.g.

4.1 Brownian motion

We start by considering a "low-pass filtered" version of a Brownian particle's trajectory. More precisely, take as the correlation time of random forces acting on the particle, and as the relaxation time of the velocity. Generally, .
So, we smooth over time intervals , subject to .
[In the frequency domain, we apply a low-pass filter with a frequency cutoff corresponding to].
We can now derive the equations of motion for the smoothed velocity. In time interval , the particle undergoes many impacts, leading to a macroscopic force. Handwavingly, we write this damping force as

Intuitively, this form arises because molecules with opposing velocity compared to the particle impart a larger force than those traveling in the same direction, hence the particle should get damped by collisions.
However, there is another, random component to the force that exists even if .
It has a large bandwidth (rapid variation in time), and occurs over timescale .
Call the random force divided by M. The correlation function of is nonzero only in an interval of width . On the much longer timescale , it is a delta function. Therefore, we write the correlation function as:

where is the spectral density at zero frequency.

With these considerations, we have the equation of motion of the Brownian particle

The RHS is a Langevin source, and it represents a source term. Since this equation is a first-order ODE, we solve it with Fourier transforms:

The spectral density of a random variable is proportional to .
So,

So the spectral density of velocity fluctuations is given in terms of the spectral density of the source, which is independent of frequency in the band of interest, .

We can get the constant for equilibrium systems using the equipartition theorem. The variance of fluctuations is

From equipartition, we know

We also have

So we find

which is the standard result for the spectral density of velocity fluctuations.

4.2 Thermal noise of an LR circuit

[van der Ziel Chap 2]

Consider an ideal LR circuit driven by .

We can solve this equation just as we did for the Brownian particle using Fourier transforms. We get

We have specified to be white, hence . Therefore, We now want to find the PSD of the source . For that we use equipartition. First compute the mean square of current fluctuations:

The energy contained in each mode, by equipartition, is then:

Therefore we again get:

In the HW you will try getting the spectral density of telegraph noise using the Langevin method.